Eur. Phys. J. D **22**, 433–440 (2003)
DOI: 10.1140/epjd/e2003-00004-2

THE EUROPEAN PHYSICAL JOURNAL D

Twin photons entangled in polarization and angular momentum

G.A. Barbosa[a]

Center for Photonic Communication and Computing, ECE Department, Northwestern University, 2145 N. Sheridan Rd., Evanston, IL 60208-3118, USA

Received 30 August 2002 / Received in final form 10 October 2002
Published online 21 January 2003 –

Abstract. Quantum states of twin photons entangled in angular momentum and polarization provide new degrees of freedom to researchers in quantum information and imaging. This work discuss these states and also emphasizes differences between two proposed models for twin photons entangled in angular momentum. Answers to the presented questions would contribute to a better understanding of this nonlinear process.

PACS. 42.50.-p Quantum optics – 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 03.67.-a Quantum information

1 Introduction

The concept of photon has brought, and will continue to bring, an indisputable richness to physics. In fact, many fundamental questions have not been answered. The distinct quantization of matter and light made the integration of Schrödinger's equation and the venerable Maxwell's equations difficult. However, unified relativistic treatments (*e.g.*, [1]) indicate that Maxwell's equations should also remain valid for single photons. The revival of interest in the fundamental properties of light beams carrying angular momentum [2,3] contributes new questions to quantum optics. Recent reviews on the angular momentum of light [4,3] give an introduction to the subject including theory and applications.

Quantum results indicate that neither component of the angular momentum (AM), $\widehat{\mathbf{L}}$ or $\widehat{\mathbf{S}}$, is a true angular momentum [5] although both describe measurable properties. Indeed, eigenvalues connected to $\widehat{\mathbf{L}}$ and $\widehat{\mathbf{S}}$ have also been shown [6] to have a continuous spectrum of eigenvalues although they sum to a discrete set of eigenvalues for the total angular momentum $\widehat{\mathbf{J}}$. A recent study, based on classical electrodynamics [7] shows (by conservation of optical angular momentum flux) that the separation of angular momentum into orbital and spin contributions is *not* an artifact of paraxial approximations as formerly believed [8]. The quantum solution of this problem has yet to be settled and, again, the photon concept will have to be invoked for a general solution. Also, the possible dependence of orbital AM on the state of motion of two distinct observers demands a relativistic treatment.

The study of quantum state entanglement using twin photons from the spontaneous parametric down conversion (SPDC) adds new zest to studies in information theory, quantum computers and cryptography. Addition of new degrees of freedom to quantum systems, such as adding orbital AM to the photon pairs [6], increases the number of dimensions of the Hilbert space needed for their description or control. An experimental study based on photon coincidences has started [9] to demonstrate transfer of AM from the pump beam to the down converted photons. Many questions have to be answered in this new field of research; this work touches on a few of these. Section 2 briefly describes SPDC and discuss some restrictions of common models to describe transfer of orbital AM to SPDC. Section 3 shows a straightforward decomposition of a light beam with orbital AM onto plane waves and stress the importance of carrying experimental verifications of the assignment [10] of an index l to the annihilation operator for downconverted photons with orbital AM. It also prepares the reader for the assignment of the l index to the down converted photon pair as a whole. Section 4 gives a fast description of the theory presented in references [6,18]. Section 5 discuss polarization and AM entanglement. Section 6 present a few ideas on possible measurements on down converted photon pairs on the position space $\{\mathbf{R}\}$, complementary to the wave vector space $\{\mathbf{K}\}$. Conclusions are then presented.

2 Spontaneous parametric down conversion and angular momentum

In SPDC [11], the workhorse for studies on state entanglement with photons, the energy and momentum constraints at the photon source establish strong temporal and spatial correlation properties between the signal (s) and idler (i) photons. These correlations establish states that cannot

[a] e-mail: barbosa@ece.northwestern.edu

be written as independent products of quantum states for s and i photons, known as entangled states. The simultaneity in pair creation leads to an *energy* entanglement of a given photon pair, in the sense that if one detects an s photon with a given energy, its conjugate i photon cannot have an arbitrary value. The emission of a single *pair*, in a random time and along two conjugated directions that cannot be known *a priori*, implies that the light state representing this process is a quantum superposition of all possible emissions. These conversions occur within a nonlinear medium, such as a $\chi^{(2)}$ crystalline medium, with quite distinct refractive indexes along different directions, or even within the amorphous glass constituting an optical fiber through a $\chi^{(3)}$ process [12]. Each class of medium defines specific phase matching conditions that must to be fulfilled for down conversion to occur.

Realistic pump beams are transversely finite and the usual focusing conditions establish a Rayleigh range z_{R}. As it will be shown, a pump beam with infinite z_{R} cannot generate down converted light with orbital angular momentum in a $\chi^{(2)}$ medium. The transversely finite condition is guaranteed by a manifold of properly weighted plane-wave modes. The helical flow of energy in light beams with angular momentum requires that the magnitude square of the field amplitude is a continuous function of the azimuthal angle on the plane transverse to the direction of light beam propagation. This can also be demonstrated in a quantum picture [6] but, in fact, this is obvious by itself. This light beam, with angular momentum l, will advance its phase by 2π for an azimuthal angular displacement of $\Delta\phi$, where $l\Delta\phi = 2\pi$. This occurs in one step s of the helix, or $s = c\Delta t$, where c is the light speed and Δt is the time interval for this advance to occur. The angular speed of the energy flow is $\Omega_\phi = \Delta\phi/\Delta t$. Substituting $\Delta t = s/c$ in Ω_ϕ gives $l\Omega_\phi = 2\pi/s$, or $s(l\nu_\phi) = c$. Detection of Ω_ϕ may demand interferometric techniques, angular momentum filters, absorption by atoms or selected molecular rotor levels and so on. The reader should look at the cited reviews for references; for example, interferometric techniques and filters have been used to show the existence of l [13]. Several experiments, including absorption of light by particulates in suspension [14], light interaction with a macroscopic mechanical rotor [15] at microwave frequency and atomic absorption [16] have demonstrated the existence of the angular momentum l associated with a light beam.

2.1 Toy models for SPDC

"Toy" model Hamiltonians used to represent twin beams may give good insights on photon entanglement in simple scattering geometries of parametric down conversion and are valuable tools to this task. However, these models usually do not consider the unit polarization vectors of the down converted beams and their product with the nonlinear susceptibility tensor of the medium. In these cases, they cannot be used to represent general cases of twin photons carrying AM because the necessary symmetry elements of the medium and the scattering geometries were not considered. These (over-)simplified Hamiltonians may lead to toy states of light of the form

$$|\Psi\rangle = \int \mathrm{d}\mathbf{q}_s \mathrm{d}\mathbf{q}_i \mathrm{d}\omega_s \mathrm{d}\omega_i \; \Phi(\mathbf{q}_s, \mathbf{q}_i; \omega_s, \omega_i) \times \hat{a}_s^\dagger(\mathbf{q}_s, \omega_s)\hat{a}_i^\dagger(\mathbf{q}_i, \omega_i)|0\rangle. \quad (1)$$

where $\Phi(\mathbf{q}_s, \mathbf{q}_i; \omega_s, \omega_i)$ is the spectral function (sometimes designate as state function, mode function and other variations) and $\mathbf{q}_j$ are the wave vectors associated with the down converted photons. Polarization vectors are usually ignored in $\Phi(\mathbf{q}_s, \mathbf{q}_i; \omega_s, \omega_i)$ and, consequently, the state becomes very simplified and usually applied to collinear cases of down conversion.

One can see the effect of this simplification without much effort by writing the unit polarization vectors of the down converted beams in a more general geometry. By doing so a dependence on the azimuthal angle ϕ (ϕ_s and ϕ_i) appears and the requirement for azimuthal symmetry shows the inconsistency of the toy models for this case. The unavoidable coupling of these vectors to the tensor $\chi^{(2)}$ shows that symmetry requirements are needed to guarantee the azimuthal symmetry for the intensity. One may then ask what are these requirements or just ignore the problem and use the toy models assuming their validity in some special cases. However, conclusions derived for particular situations should not be extended to a general case because they may not be valid; the crystal symmetry may not even allow such process to occur. An introduction to this problem can be seen in references [6,17,18]. These considerations are often ignored in the literature, or not believed to exist, *e.g.* [3,19]. Even phase relationships between pump and down-converted beams arise precisely from the response function ($\chi^{(2)}$ tensor) of the nonlinear medium. This response function defines the virtual absorption of the incident field and the fast following emission (or scattering) of the medium. This process is what defines if the medium will absorb $z-$polarized photons, say, and emit them with an orthogonal polarization to the pump beam as in the type I down conversion or with both polarizations as in a type II scattering. This process also imprints the phase relationship carried by the pump mode to the down converted photon pair. A common understanding should be built on these basic points.

3 Single photons carrying angular momentum l. Some questions

Although no one questions mode decompositions in angular momentum cases, one may wonder if the phase structure associated with the mode energy would imply that each photon carries a quantum of orbital angular momentum with respect to a quantization axis. In order to understand this problem in terms of the usual plane wave description, one could start with a finite mode. A light mode, limited in the direction transverse to the propagation direction, implies that a range of wave vectors are contributing to establish the mode finiteness. These wave

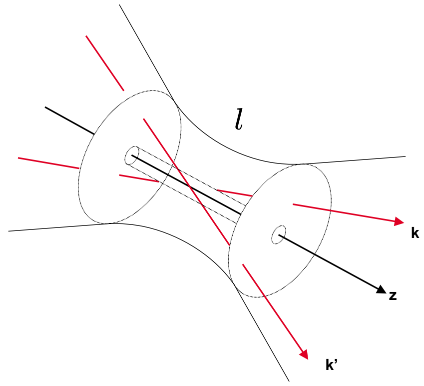

Fig. 1. Wave vectors **k** and **k**' around the focus in a mode with angular momentum l; they do not cross the $z-$axis. Wave vectors can be selected by spaced pinholes, for example.

vectors, associated with phase relationships, establish the helical propagation of the light energy around the quantization axis.

The quantized electric field describing a beam with orbital angular momentum l can be written as

$$\widehat{\mathbf{E}}_l(\mathbf{r},t) = - \mathrm{i}\sum_{\mathbf{k}} \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon V}} \left[\mathbf{u}_{\mathbf{k},l}(\mathbf{r})\ \widehat{a}_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}t} - \mathbf{u}^*_{\mathbf{k},l} \widehat{a}^+_{\mathbf{k}}\ \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}t}\right], \quad (2)$$

where $\mathbf{u}_{\mathbf{k},l}(\mathbf{r})$ are classical solutions given by Maxwell's equations for specific geometries (wave guides, lenses, cavities etc.) and $\widehat{a}_{\mathbf{k}}$ and $\widehat{a}^\dagger_{\mathbf{k}}$ define the quantum statistical properties associated to the excitations of the mode. Figure 1 sketches a few eigenvectors associated to an angular momentum mode. As phase is always associated with a given reference, one may wonder if a direct detection (energy) of a single photon, from a mode with a given phase structure, could reveal the angular momentum l. Interferometric techniques exploring probability amplitudes reveal variations in intensity that are associated to the angular momentum l. Macroscopic averages over these events lead to forked diagrams or to the mode pictures presented in [20]. As interference is a phenomenon determined at the single photon level, these results could be understood as a comparison between single photons registered at distinct positions. In other words, the angular momentum l can be revealed by comparing photon registers with distinct wave vectors. Let us look at the decomposition of a vector $\mathbf{u}_l(\mathbf{r},t)$ describing a Laguerre-Gaussian wave packet into a plane wave basis $\hat{\epsilon}_{\mathbf{k}}$ (the index l can be used to designated (p,l) in a Laguerre mode):

$$\mathbf{u}_l(\mathbf{r},t) = \sum_{\mathbf{k}} \mathbf{U}_{\mathbf{k},l}(\mathbf{r})\hat{\epsilon}_{\mathbf{k}}\ \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}t}, \quad (3)$$

where $\mathbf{U}_{\mathbf{k},l}$ are components of the unitary matrix U $(U^{-1} = U^\dagger)$ that transforms the Laguerre-Gaussian to a plane wave basis. The quantized electric field,

$$\widehat{\mathbf{E}}^{(+)}(\mathbf{r},t) = \sum_{\mathbf{k}} l_E(\mathbf{k})\hat{\epsilon}_{\mathbf{k}}\ \widehat{a}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}, \quad (4)$$

where $l_E(\mathbf{k}) = -\mathrm{i}\sqrt{\hbar\omega_{\mathbf{k}}/(2\epsilon V)}$, can be written using the identity $\widehat{a}_{\mathbf{k}} = \sum_{\mathbf{k}'} \delta_{\mathbf{k},\mathbf{k}'}\widehat{a}_{\mathbf{k}'}$ and $\sum_l U_{l\mathbf{k}}U^*_{\mathbf{k}'l} = \delta_{\mathbf{k},\mathbf{k}'}$; one obtains

$$\widehat{\mathbf{E}}^{(+)}(\mathbf{r},t) = \sum_l \left[\sum_{\mathbf{k}'} U^*_{\mathbf{k}'l}\widehat{a}_{\mathbf{k}'}\right]\left[\sum_{\mathbf{k}} U_{l\mathbf{k}} l_E(\mathbf{k})\hat{\epsilon}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}\right]$$
$$\equiv \sum_l \widehat{c}_l \mathbf{u}_l(\mathbf{r},t) = \sum_l \widehat{\mathbf{E}}^{(+)}_l(\mathbf{r},t), \quad (5)$$

where $\widehat{c}_l = \sum_{\mathbf{k}'} U^*_{k'l}\widehat{a}_{\mathbf{k}'}$ is the annihilation operator for a quantum of excitation in the Laguerre-Gaussian mode. $\widehat{c}^\dagger_l$ creates a set of photons with wave vectors **k** and $U_{l\mathbf{k}}$ is the probability amplitude to get a photon with wave vector **k** in mode l. In the plane wave or momentum representation the annihilation operators $\widehat{a}_{\mathbf{k}}$ do *not* carry the variable l. This lack of a signature of AM in a single photon, with a defined or selected wave vector, indicates that in this case *no* measurement would show an l signature.

An interesting method, proposed in reference [20], intends to identify the angular momentum carried by single photons. It clearly identify distinct l-modes, as it was correctly shown in that reference. Without selecting a wave vector, a single photon detection could be seen originating from one among all possible wave vectors present in the l-beam. In a sense, as defined by the l filter, the probability amplitudes for all possible wave vectors in the beam could interfere, resulting in the single event connected to the value l. One may start asking if a single photon passing through the filter system would be able to excite an atomic or molecular system with the selection rule adequate for the specific l-transition or, say, to turn an atomic or molecular system by light absorption with a resulting motion corresponding to the given l-value. The proposed method is still to be demonstrated experimentally, at the single photon level. If correct, it would provide a demonstration for the association of the index l to the annihilation and creation operators for photons carrying orbital angular momentum.

4 Entangled angular momentum states and constant of motion

The question whether angular momentum from a pump beam could be transferred to photon pairs from SPDC was studied theoretically in [6,18,22] and was experimentally demonstrated using type I down conversion in reference [9]. These studies agree on the fact that angular momentum transfer is possible and that the angular momentum value l should be attached to the photon pair as a whole. Experimental results in [9] show correlation in photon counts, in type I down conversion, whenever angular momentum filters placed in front of the s and i detectors are chosen with a value that adds up to the angular momentum per photon of the pump beam.

However, a fundamental disagreement exists between the two theoretical treatments: (1) references [6,18] emphasizes that the nonlinear response function of the material, given by the tensor $\chi^{(2)}$, determines the coupling between the pump polarization vector and the down converted polarization vectors at chosen (arbitrary) propagation angles. As the *symmetry* of the crystal is reflected in

$\chi^{(2)}$, and on the linear refractive properties of the medium, the scattering efficiency depends on the symmetry, in general. It also shows that the quantum state of light representing down conversion processes has to predict a scattering intensity independent of the azimuthal angle ϕ to guarantee energy flow conservation on the transverse plane to the pump propagation direction. The probability amplitude for down conversion to occur is directly proportional to the product of $\chi^{(2)}$ and the unitary polarization vectors. Consequently, azimuthal independence is not guaranteed for arbitrary scattering geometries. It is also stressed that, in general, this is very relevant at large scattering angles. At low scattering angles (paraxial cases) this condition could be relaxed in some cases because the unitary polarization vectors may present a negligible dependence on ϕ. (2) Reference [22], by its turn, emphasizes that crystal symmetry plays *no* role in determining the orbital angular momentum of any of the beams in general, and present a theory adequate for paraxial cases.

The predicted differences are too striking to be ignored. Experimental tests could probe for the predicted differences. For example, if an experiment designed to reveal a clear azimuthal angular dependence in the coupling $\chi^{(2)}$-to-polarization vectors succeeds to demonstrate down conversion, the model [6,18] will be shown wrong, in favor of [22]; and *vice versa*.

Reference [6] also predicts that the l-entanglement may leave its signature on a photon pair, without relying in mode decompositions as was done in reference [9]. This section explains this idea, after introducing the basic notations.

An interaction Hamiltonian describing parametric down conversion and that contains the minimum necessary features to understand features beyond quasi-collinear or paraxial cases is

$$\begin{aligned}\widehat{H}_I =& \sum_{\sigma,\sigma'}\int \mathrm{d}^3k_s\, \mathrm{d}^3k_i\, l_E^{(*)}(\omega_{\mathbf{ks}})\, l_E^{(*)}(\omega_{\mathbf{k}i})\, \widehat{a}^{\dagger}(\mathbf{k}_s,\sigma)\, \widehat{a}^{\dagger}(\mathbf{k}_i,\sigma')\\ &\times \mathrm{e}^{\mathrm{i}(\omega_{\mathbf{k_s}}+\omega_{\mathbf{k}i})t}\chi_{qjk}\left(\mathbf{e}_{\mathbf{ks},\sigma}\right)_j^*\left(\mathbf{e}_{\mathbf{k}i,\sigma'}\right)_k^*\\ &\times \int_{V_I} \mathrm{d}^3r\, \mathbf{E}_{Pq}(\mathbf{r},t)\, \mathrm{e}^{-\mathrm{i}(\mathbf{k}_s+\mathbf{k}_i)\cdot\mathbf{r}} + \mathrm{h.c.} \end{aligned}\tag{6}$$

V_I is the nonlinear interaction volume, $\mathbf{E}_P(\mathbf{r},t)$ is the electrical field associated with the pump beam, subscripts s and i indicate signal and idler and indexes (σ,σ') represent possible polarization states, $l_E^{(*)}(\omega_{\mathbf{k}}) = -\mathrm{i}\sqrt{\hbar\omega_{\mathbf{k}}/2\epsilon(\mathbf{k},\sigma)}$. Repeated indexes q, j and k imply summations. The pump beam is a LG mode propagating along $\widehat{z}$ and polarized along direction $\widehat{x}(=\mathbf{e}_1 \to q)$:

$$\begin{aligned}&\mathbf{E}_{l,p}(\rho,\phi,z;t) \equiv \psi_{lp}(\mathbf{r})\, \mathrm{e}^{\mathrm{i}(k_Pz-\omega_Pt)}\,\widehat{x}\\ &= \frac{A_{lp}}{\sqrt{1+(z^2/z_{\mathrm{R}}^2)}}\left(\frac{\sqrt{2}\rho}{w(z)}\right)^l L_p^l\left(\frac{2\rho^2}{w^2(z)}\right)\exp\left(\mathrm{i}\frac{k_P\rho^2}{2q(z)}\right)\\ &\times \mathrm{e}^{\mathrm{i}l\arctan(y/x)}\exp\left(-\mathrm{i}(2p+l+1)\arctan\frac{z}{z_{\mathrm{R}}}\right)\\ &\times \mathrm{e}^{\mathrm{i}(k_Pz-\omega_Pt)}\,\widehat{x}, \end{aligned}\tag{7}$$

where z_{R} is the Rayleigh length, $w(z) = w_0\sqrt{1+z^2/z_{\mathrm{R}}^2}$ (w_0 is the beam radius at the waist $z=0$), $q(z) = z - \mathrm{i}z_{\mathrm{R}}$ and $\rho = \sqrt{x^2+y^2}$. The first order approximation for the state vector of the down-converted light leads to

$$\begin{aligned}|\psi(t)\rangle =& |0\rangle + \sum_{\sigma,\sigma'}\int \mathrm{d}^3k_s \int \mathrm{d}^3k_i\, \chi^{(2)}{}_{qjk}\left(\mathbf{e}_{s,\sigma}\right)_j^*\left(\mathbf{e}_{i,\sigma'}\right)_k^*\\ &\times l_E^{(*)}(\omega_{\mathbf{ks}})\, l_E^{(*)}(\omega_{\mathbf{k}i})\, \mathrm{e}^{\mathrm{i}\Delta\omega\, t/2}\, T(\Delta\omega)\\ &\times \widetilde{\psi}_{lp}(\Delta\mathbf{k})\, \widehat{a}^{\dagger}(\mathbf{k}_s,\sigma)\, \widehat{a}^{\dagger}(\mathbf{k}_i,\sigma')\,|0\rangle, \end{aligned}\tag{8}$$

where $\Delta\omega = \omega_{\mathbf{ks}} + \omega_{\mathbf{k}i} - \omega_P$, $\Delta\mathbf{k} = \mathbf{k}_s + \mathbf{k}_i - \mathbf{k}_P$, $\widetilde{\psi}_{lp}(\Delta\mathbf{k}) = \int_{V_I}\mathrm{d}^3r\,\psi_{lp}(\mathbf{r})\exp(-\mathrm{i}\Delta\mathbf{k}\cdot\mathbf{r})$ and the time window function is $T(\Delta\omega) = \sin(\Delta\omega\, t_{\mathrm{int}}/2)/(\Delta\omega/2)$. Under reasonable approximations [6] one has

$$\begin{aligned}\widetilde{\psi}_{lp}(\Delta\mathbf{k}) = B_{lp}\mathrm{e}^{\mathrm{i}\Delta k_z z_0/2}\, W(\Delta k_z)\,\varphi_{lp}(\rho_s,\rho_i,\phi_s-\phi_i)&\\ \times \rho_k^l\, \mathrm{e}^{\mathrm{i}\gamma_k l},&\end{aligned}\tag{9}$$

where z_0 is the position of the center of the nonlinear medium, $W(\Delta k_z) = \sin(\Delta k_z\ell_{\mathrm{c}}/2)/(\Delta k_z/2)$ is the spatial window function and ℓ_{c} is the crystal length, ϕ_s and ϕ_i are the azimuthal angles for signal and idler photons, $\rho_s = k_s\sin\theta_s$ and $\rho_i = k_i\sin\theta_i$ are transverse components of the wave vectors. One should observe from equation (9) that for $\rho_k = 0$, $\rho_k^l = 0$ unless $l = 0$. Also,

$$\begin{aligned}B_{lp} &= \left(z_{\mathrm{R}}\pi/k_P^{l+1}\right)\left(z_{\mathrm{R}}\sqrt{2}/w_0\right)^l\\ &\quad\times\exp\left[-\frac{\pi}{2}\mathrm{i}(1-l-p)\right]2^{p-l+1},\\ \rho_k &= \sqrt{\rho_s^2+\rho_i^2+2\rho_s\rho_i\cos(\phi_s-\phi_i)},\\ \gamma_k &= \arctan\frac{\rho_s\sin\phi_s+\rho_i\sin\phi_i}{\rho_s\cos\phi_s+\rho_i\cos\phi_i},\end{aligned}$$

and

$$\begin{aligned}&\varphi_{lp}(\rho_s,\rho_i,\phi_s-\phi_i) =\\ &\qquad L_p^l\left(\frac{z_{\mathrm{R}}\,\rho_k^2}{k_P}\right)\exp\left(-\frac{z_{\mathrm{R}}\,\rho_k^2}{2k_P}\right)\exp\left(-\mathrm{i}\frac{z_0\,\rho_k^2}{2k_P}\right)\cdot\end{aligned}\tag{10}$$

4.1 Entangled angular momentum states

At phase matching condition, the state of light derived from equation (8) was found to be

$$\begin{aligned}|\psi_{lp}(t)\rangle =& \int \mathrm{d}^3k_s\mathrm{d}^3k_i\sum_{n=0}^{l} F_{lp}^{(0,n)}(\mathbf{k}_s,\mathbf{k}_i)\\ &\times |1_{\mathbf{k}_s}, j_z = n\rangle\, |1_{\mathbf{k}_i}, j_z = l-n\rangle \equiv \sum_{n=0}^{l}|\Psi_n\rangle,\end{aligned}\tag{11}$$

where the amplitude probability $F_{lp}^{(0,n)}(\mathbf{k}_s,\mathbf{k}_i)$ is

$$\begin{aligned}F_{lp}^{(0,n)}(\mathbf{k}_s,\mathbf{k}_i) =& A_{\mathbf{k}_s;\mathbf{k}_i}\, l_E^{(*)}(\omega_{\mathbf{k}_s})\, l_E^{(*)}(\omega_{\mathbf{k}_i})\, B_{lp}\rho_k^l\\ &\times \mathrm{e}^{\mathrm{i}\gamma_k l}\, \mathrm{e}^{\mathrm{i}\Delta\omega\, t/2}\, T(\Delta\omega)\, W(\Delta k_z)\binom{l}{n}\varphi_{lp}(\rho_s,\rho_i,\phi_s-\phi_i),\end{aligned}$$

and

$$A_{\mathbf{k}_s;\mathbf{k}_i} = \chi^{(2)}{}_{qjk} \left[(\mathbf{e}_{\mathbf{k}_s})_j^* (\mathbf{e}_{\mathbf{k}_i})_k^* + (\mathbf{e}_{\mathbf{k}_i})_j^* (\mathbf{e}_{\mathbf{k}_s})_k^* \right] \cdot \quad (12)$$

Indexes σ and σ' are included in the wave vectors. One could write equation (11) as

$$|\psi_{lp}(t)\rangle = \int \mathrm{d}^3 k_s \mathrm{d}^3 k_i \sum_{n=0}^{l} F_{lp}^{(0,n)}(\mathbf{k}_s, \mathbf{k}_i) \times \widehat{a}^\dagger_{\mathbf{k}_{s,n}} \widehat{a}^\dagger_{\mathbf{k}_{i,l-n}} |0_s\rangle |0_i\rangle . \quad (13)$$

Each "biphoton" created by $\widehat{a}^\dagger_{\mathbf{k}_{s,n}} \widehat{a}^\dagger_{\mathbf{k}_{i,l-n}}$, with probability amplitude $F_{lp}^{(0,n)}(\mathbf{k}_s, \mathbf{k}_i)$ carries information on the angular momentum l. What produces a difference between this case and the one treated in Section 3 are the phase matching conditions. The simplified approach used for phase matching in reference [18] for the case $(p = 0, l)$ gave, respectively, the longitudinal and transverse conditions

$$-k_P + (k_s \cos\theta_s + k_i \cos\theta_i) = \frac{2\pi}{\ell_{\mathrm{c}}}, \quad (14)$$

$$\rho_s^2 + \rho_i^2 + 2\rho_s\rho_i \cos(\phi_s - \phi_i) = \frac{k_P\, l}{z_{\mathrm{R}}} \cdot \quad (15)$$

One can see that making $z_{\mathrm{R}} \to \infty$ is equivalent to $l \to 0$. This is sufficient to show that without beam divergence, orbital angular momentum will not be transferred to a conjugated photon pair. Of course, for finite but large z_{R} the signature of AM transfer may become vanishingly small to be detected. This can also be seen from equations (9, 10), where $\varphi_{l0}(\rho_s, \rho_i, \phi_s - \phi_i) \to 0$ when $z_{\mathrm{R}} \to 0$.

Using this simplified approach the correlated emissions of signal and idler on the transverse wave vector plane was obtained, as shown in Figure 2. In reference [18] it was shown that for Laguerre $(l, 0)$ modes, the condition for maximum of the probability amplitude, a phase matching condition, gives $\rho_{k_0} = \sqrt{k_P l / z_{\mathrm{R}}}$. This condition is equivalent to the "constant of motion"

$$|(k_{sx} + k_{ix})\widehat{x} + (k_{sy} + k_{iy})\widehat{y}|^2 = \frac{k_P l}{z_{\mathrm{R}}} \cdot \quad (16)$$

With the knowledge of the parameters k_P and z_{R} a measurement of the wave vectors of the down converted photons by an observer stationary with respect to the laser beam would reveal the angular momentum l of the LG pump beam. This constraint appear due to the transverse phase matching and, this way, the conjugated pair of photons carry the variable l. In a way, one of the photons in the pair acts as a reference for the other or, in other words, the l information is "written" on the two photons due to the entanglement. The angular momentum l is then a variable attached to the photon pair. Probing just one photon of the pair will not reveal the l-mode information in the pump beam.

All of these results were based on SPDC produced by CW monochromatic pump beam. Angular momentum entanglement could be also studied in the case of a broad band pump [21] where the uncertainty in the down conversion emission times is reduced and the requirements for AM entanglement may be modified.

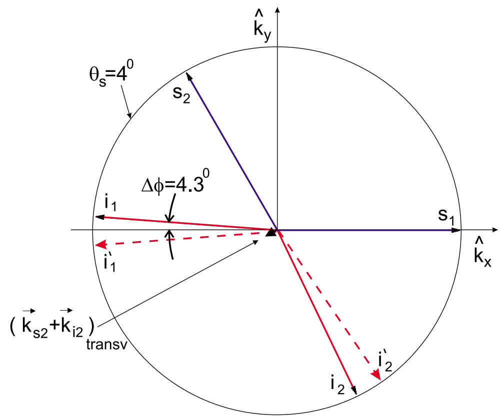

Fig. 2. Conjugated wave vectors on the transversal plane $(\hat{k}_x, \hat{k}_y)$. Solid lines at each (s_j, i_j) indicate a possible wave vector for conjugated signal and idler. Dashed lines indicate another emission possibility for each counterpart wave vector given by a solid line. The emission cone has an angular aperture of 4° (small angle regime). Each conjugate wave vector is displaced 4.3° (or −4.3°) from the value π. A sum of the wave vectors $\mathbf{k}_{s2} + \mathbf{k}_{i2}$ is indicated. Every sum of conjugated wave vectors has identical magnitude.

4.2 Constraints imposed by the crystal symmetry on the down conversion process

In Section 2.1 the importance of geometry on the down conversion process was mentioned. This dependence is contained in $A_{\mathbf{k}_s,\sigma;\mathbf{k}_i,\sigma'}$, as given by equation (12). It contains the scattering angles for specific down conversion geometries and the polarization vectors; the crystal symmetry is embedded in the susceptibility tensor and in the polarization vectors as given by Fresnel's and Sellmeir's equations. For cases where $A_{\mathbf{k}_s,\sigma;\mathbf{k}_i,\sigma'}$ does not depend on the azimuthal angles, or this dependence can be neglected, one should expect transfer of angular momentum from the pump beam to the down converted photons [6] (see also discussions in [17,23]). For example, in quasi-collinear scattering in type I down conversion the amplitude $A_{\mathbf{k}_s,\sigma;\mathbf{k}_i,\sigma'}$ has an almost negligible dependence on the azimuthal angle; AM entanglement has been demonstrated for this case [9]. In type II down conversion, the polarization vectors are more dependent on the azimuthal angle, even at low scattering angles, due to the propagation of down converted light polarized along the extraordinary direction of the crystal. Experimental studies of AM entanglement in type II down conversion are, in particular, expected to give important results for comparison with existing theoretical descriptions.

4.3 Rotating reference

The orbital angular momentum may depend on the reference utilized for its description. For example, looking at the phase structure defining an l-mode one may see this signature by measuring the phase shift as the beam is rotated by a given angle [24] on a table-top experiment.

In the case of down converted photons, looking at the transverse scattering plane (see Fig. 2), at radius ρ_{k_0}, one may measure the angle $\Delta\phi$ deviated from π. One may ask what value would be seen by an observer rotating, with an arbitrary speed, around the pump beam propagation axis (quantization axis). In other words, one may ask if a rotating observer could measure a distinct value l' for the angular momentum. Distinct measurements could lead to a gyroscopic device to detect the relative motion between two separated observers, in a different geometry than the one described in reference [24]. *E.g.*, an observer on Earth could detect $\Delta\phi$ while another observer, on a spaceship, may detect $\Delta\phi'$, from which the relative frequency between the two observers could be obtained.

An estimate for l' can be obtained by following a phase value during a time interval Δt, obtaining an arc s_a at fixed ρ_{k_0}: $s_a = \Omega_\phi \rho_{k_0} \Delta t$. Considering the arc s_a as a rectilinear section, as a simplification, one could say that for an observer rotating with angular speed ω_r around the z-axis (paraxial scattering, for simplicity) the measured arc s'_a will be

$$s'_a = \gamma\,[s_a - (\omega_r \rho_{k_0})\Delta t] = \gamma \rho_{k_0} \Delta t(\Omega_\phi - \omega_r), \qquad (17)$$

where $\gamma = 1/\sqrt{1-(\omega_r\rho_{k_0}/c)^2}$. Thus $\Delta s_a = s_a - s'_a = \rho_{k_0}\Delta t[\omega_r - \Omega_\phi(\gamma-1)]$, from which a change $\delta\phi = \rho_{k_0}\delta s_a$ from $\Delta\phi$ measured by the stationary observer results and may reveal l'.

Processes relying in multi-photon-pairs entangled in angular momentum may provide an enhanced sensitivity for detection of relative motion based on photon coincidences.

5 Polarization and angular momentum entanglement

The addition of polarization entanglement to the s and i system has produced several studies. The equations describing the generation and propagation of the down converted light in matter prescribe states of light with defined polarization states, as allowed by $A_{\mathbf{k}_s,\sigma;\mathbf{k}_i,\sigma'}$. In this way, a certain arrangement can generate light polarized in a single polarization state, say horizontal (H) for both s and i photons (type I down conversion) or, H for a s photon and vertical (V) for the i photon (type II down conversion). For type II and along special directions [25,26], if one knows the polarization of a detected photon one also knows that entanglement defines the polarization of the conjugated photon. These cases are produced by states of down-converted light that are either (H_s, V_i) or (V_s, H_i) with the same probability. These entangled states are akin to the spin states associated to an electron pair, and bring with them the same conceptual problems associated with the famous EPR paradox. If one associates a possible bit with each polarization, the normalized state of the two particles propagating along directions a and b is written in the symmetric form

$$|\Psi^{(+)}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_a|V\rangle_b + |V\rangle_a|H\rangle_b\right). \qquad (18)$$

As a bosonic system the symmetry of the state has to be conserved under "spin" (polarization) and position exchange. This state can be interpreted as carrying two bits of information, or two polarization states, for each photon [27]. This is a result that cannot be reproduced by a classical system. One should understand that, by using a polarizer, the measurement process projects the state to a specific polarization for each photon (1 bit/photon) but the fundamental uncertainty on the polarization state resides until the specific measurement is performed. A complete Bell state bases can be obtained by appropriate design of the emitter system and complementary optics.

Type I down conversion has also been used to produce entangled states of the form given by equation (18) [28]. However, in this case one should rely on trajectory superpositions and neglect photon pairs in a same trajectory; only then entanglement results. In type II down conversion the entangled state is defined at the source while in type I it occurs due to the neglect of a set of photon occurrences; one may classify this last case as a pseudo-entanglement or, even more drastically, as an incomplete entanglement. Nevertheless, one may argue that when properly normalized to take into account the set of ignored events, type I down converted states may be useful to reveal entanglement characteristics for AM studies.

The richness of adding angular momentum to the parametric down conversion processes have been indicated theoretically by the first time in [6], and experimentally in [9], and can be represented by equation (13). Although these ideas were to be expected from basic concepts, and in fact they have been latter studied in [29], exploring these new features is not a straightforward task [18,22]. Figure 3 sketches a possible arrangement to achieve angular momentum and polarization entanglement by path indistinguishability of entangled signal and idler photons in type I down conversion. In this entanglement configuration, coincident photons from the two BS ports will be described by

$$|\Psi_n\rangle = \frac{1}{2}\left(\,|H,l-n\rangle_a|V,n\rangle_b + |V,n\rangle_a|H,l-n\rangle_b \right.$$
$$\left. +|V,l-n\rangle_a|H,n\rangle_b + |H,n\rangle_a|V,l-n\rangle_b\,\right),\ \ (n=0,\cdots,l),$$

where $|\Psi_n\rangle$ was defined in equation (11).

One may question if polarization and AM entanglement could occur using type II down conversion instead of type I. A study of the term $A_{\mathbf{k}_s,\sigma;\mathbf{k}_i,\sigma'}$ may be necessary to provide an answer for this question.

6 AM entanglement in R-space

Frequently, studies of parametric down conversion explore properties in the wave vector space $\{\mathbf{k}_s, \mathbf{k}_i\}\cdot$ How to extend these studies to the position space $\{\mathbf{r}_s, \mathbf{r}_i\}$, without using imaging techniques? Of course, an angle defining a wave vector propagation will coincide with an angle obtained from position measurements in photon detection but variances in $\{\mathbf{k}_s, \mathbf{k}_i\}$ and $\{\mathbf{r}_s, \mathbf{r}_i\}$ spaces provide complementary information. In principle, this is also an

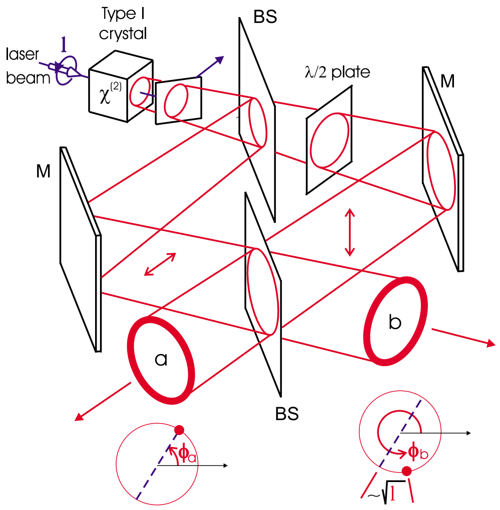

Fig. 3. Basic arrangement for polarization and angular momentum entanglement. Down converted beams are splitted at beam splitter BS and recombined after one of the beams passes through a half-wave plate. Other mirrors (M) can be included as necessary. A dychroic mirror separates the pump photons from the down converted pairs right after the crystal.

adequate frame to study the problem of different observers in relative motion.

Using the wave state for twin photons with angular momentum, equation (11), written in a wave vector bases, a Fourier transform could be applied leading to

$$
\begin{aligned}
|\psi(t)\rangle &= \sum_n \sum_{\mathbf{k}_s} \sum_{\mathbf{k}_i} F^{(n)}(\mathbf{k}_s,\mathbf{k}_i)|1_{\mathbf{k}_{s,n}}\rangle\,|1_{\mathbf{k}_{i,l-n}}\rangle \\
&= \left[\frac{V}{(2\pi)^3}\right]^2 \sum_n \int \mathrm{d}\mathbf{k}_s \int \mathrm{d}\mathbf{k}_i F^{(n)}(\mathbf{k}_s,\mathbf{k}_i)|1_{\mathbf{k}_{s,n}}\rangle|1_{\mathbf{k}_{i,l-n}}\rangle \\
&= \frac{1}{V^2}\sum_n \int \mathrm{d}\mathbf{r}_s \int \mathrm{d}\mathbf{r}_i\, \Gamma^{(n)}(\mathbf{r}_s,\mathbf{r}_i)\,|1_{\mathbf{r}_{s,n}}\rangle\,|1_{\mathbf{r}_{i,l-n}}\rangle, \qquad (19)
\end{aligned}
$$

for $\Gamma^{(n)}(\mathbf{r}_s,\mathbf{r}_i) = \Gamma^{(n)}(-\mathbf{r}_s,-\mathbf{r}_i)$ in the medium. This is easily accomplished by a laser focused at the center of the nonlinear medium. The probability amplitude to find a conjugated pair in $\mathbf{r}_s$ and $\mathbf{r}_i$ is associated with the *spatial* function $\Gamma(\mathbf{r}_s,\mathbf{r}_i)$.

Fourier transforming the probability amplitude in the wave vector space as described in equation (19) gives directly the amplitude $\Gamma(\mathbf{r}_s,\mathbf{r}_i)$ in series solution. The first terms in this series, for a given (λ_s,λ_i) (or k_s,k_i), at the phase matching condition, is

$$
\begin{aligned}
&\Gamma^{(n)}_{l0}(\mathbf{r}_s,\mathbf{r}_i) = \mathrm{e}^{\mathrm{i}\,k_s(z_i+z_s)\cos\theta_0} {k_s}^4 \sin^2\theta_0 a_{lp}\, l_{i2}\, l_{s2}\, {\rho_{k_0}}^l B_{pl} \\
&\times \binom{l}{n} L_p^l\!\left(\frac{{\rho_k}^2\,{\omega_0}^2}{2}\right)\mathrm{e}^{-\frac{z_{\mathrm{R}}\,{\rho_{k_0}}^2}{2\,k_p}}\left[\mathrm{e}^{\mathrm{i}l\delta/2}J_0(a_p)+\mathrm{e}^{-\mathrm{i}l\delta/2}J_0(a_n)\right]
\end{aligned}
$$

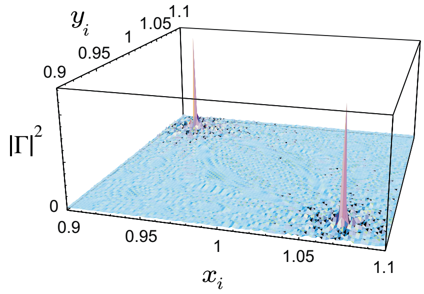

Fig. 4. Probability $|\Gamma|^2$ (arbitrary units) for idler detection in the (x,y) plane once a signal photon is detected at ($x_s = -1$, $y_s = -1$), for the case ($p = 0$, $l = 1$).

where

$$
\begin{aligned}
a_p &= k_s \sin\theta_0 \\
&\times\sqrt{(x_s - x_i\cos\delta - y_i\sin\delta)^2 + (y_s - y_i\cos\delta + x_i\sin\delta)^2} \\
a_n &= k_s\sin\theta_0 \\
&\times\sqrt{(x_s - x_i\cos\delta + y_i\sin\delta)^2 + (y_s - y_i\cos\delta - x_i\sin\delta)^2},
\end{aligned}
$$

and $\delta = |\phi_s - \phi_i|$ gives the phase difference according to the phase matching condition (here $l = |l|$ and two possible events are shown). These first terms in the series would describe twin photons with low values of angular momentum. Figure 4 shows the expected idler occurrences over the transverse (x_i, y_i) plane when a signal photon was detected at position $(x_s = -1, y_s = -1)$ in cm. The deviation from $(x_i = y_i = 1)$ cm shows the off-plane occurrence for signal-idler-pump photons. Of course, this was obtained under several simplifications but the off-plane deviation is clear. The (x,y) and z-dependencies are tied up by the scattering angle θ_0.

6.1 Quasi-probability distributions

The Wigner distribution has been established as a standard one to which other quantum distributions could be compared and a Wigner function for this twin photon system entangled in angular momentum could eventually been measured, with quadratures of the scattered field providing the variables for the distribution. However, other quasi-probability functions can be useful to provide information on quantum or classical systems. Reference [30] presents a Wigner representation of Laguerre-Gaussian beams and derives connections between this and the Wigner representation of Hermite-Gaussian modes.

For the case of AM entanglement of photon pairs a quasi-probability distribution can be written to give the probability of occurrence of specified wave vectors ($\mathbf{k}$) and positions ($\mathbf{q}$), for *signal* and *idler* pair, through

$$
\begin{aligned}
W_c(\mathbf{q}_s,\mathbf{k}_s;\mathbf{q}_i,\mathbf{k}_i) &= \left(\frac{1}{2\pi}\right)^6 \iint \mathrm{d}\mathbf{r}_s \mathrm{d}\mathbf{r}_i\, \mathrm{e}^{\mathrm{i}\mathbf{k}_s\cdot\mathbf{r}_s}\mathrm{e}^{\mathrm{i}\mathbf{k}_i\cdot\mathbf{r}_i} \\
&\times\left\langle \mathbf{q}_s - \frac{\mathbf{r}_s}{2}, \mathbf{q}_i - \frac{\mathbf{r}_i}{2}\right|\widehat{\rho}\left|\mathbf{q}_s + \frac{\mathbf{r}_s}{2}, \mathbf{q}_i + \frac{\mathbf{r}_i}{2}\right\rangle. \qquad (20)
\end{aligned}
$$

Writing $\widehat{\rho} = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is given by equation (19) results

$$W_c = \left(\frac{1}{2\pi}\right)^6 \iint \mathrm{d}\mathbf{r}_s \mathrm{d}\mathbf{r}_i \, \mathrm{e}^{\mathbf{i}\mathbf{k}_s\cdot\mathbf{r}_s} \mathrm{e}^{\mathbf{i}\mathbf{k}_i\cdot\mathbf{r}_i} \times \Gamma^*\left(\mathbf{q}_s + \frac{\mathbf{r}_s}{2}, \mathbf{q}_i + \frac{\mathbf{r}_i}{2}\right) \Gamma\left(\mathbf{q}_s - \frac{\mathbf{r}_s}{2}, \mathbf{q}_i - \frac{\mathbf{r}_i}{2}\right) \cdot \quad (21)$$

Equation (21) satisfies the basic properties for a quasi-probability function including

$$\iint \mathrm{d}\mathbf{k}_s \mathrm{d}\mathbf{k}_i W_c(\mathbf{q}_s, \mathbf{k}_s; \mathbf{q}_i, \mathbf{k}_i) = |\Gamma(\mathbf{q}_s, \mathbf{q}_i)|^2. \quad (22)$$

Projections of this quasi-probability function W_c on the wave vector or R space can also be used to predict results of experiments with photons in angular-momentum entangled states, as exemplified in Figure 4.

7 Conclusions

Recently, the transfer of AM from a pump beam to down converted photons has been demonstrated for type I down conversion and in a low scattering angles [9]. This work proposes a few ideas connected to this process and show fundamental divergences between two published theories describing AM transfer. Basically, one of them, references [6,18], stress a dependence with the crystal geometry. This dependence appears in some terms, including the tensorial product of the nonlinear susceptibility and the unitary polarization vectors. At low small scattering angles this dependence can be relaxed for some crystals but it is more visible at large scattering angles. The other theory, described in references [19,22], takes the point of view that the orbital angular momentum transfer is not dependent on the discussed tensorial product, that is related to the crystal symmetry and to the unitary polarization vectors for arbitrary scattering angles.

Studies on AM transfer may depend on a correct model for further advances. The reader is invited to join the discussion.

G.A. Barbosa acknowledges support from NSF grant PHY-0219382 and DARPA grant F30602-01-2-0528. Thanks are also due to Paul Voss for a critical reading of the manuscript and to Prem Kumar for interesting discussions.

References

1. A. Gersten, Found. Phys. Lett. **12**, 291 (1999)
2. R.A. Beth, Phys. Rev. **48**, 471 (1935); *ibid.* **50**, 115 (1936)
3. L. Allen, J. Opt. B: Quant. Semiclass. Opt. **4**, S1 (2002)
4. L. Allen, M.J. Padgett, M. Babiker, Progr. Opt. **XXXIX**, 291 (1999)
5. S.J. van Enk, G. Nienhuis, J. Mod. Opt. **41**, 963 (1994)
6. H.H. Arnaut, G.A. Barbosa, Phys. Rev. Lett. **85**, 286 (2000)
7. S.M. Barnett, J. Opt. B: Quant. Semiclass. Opt. **4**, S7 (2002)
8. S.M. Barnett, L. Allen, Opt. Commun. **110**, 670 (1994)
9. A. Mair, A. Vaziri, G. Weihs, A. Zeilinger, Nature **412**, 313 (2001)
10. L.C. Dávila Romero, D.L. Andrews, M. Babiker, J. Opt. B: Quant. Semiclass. Opt. **4**, S66 (2002)
11. W.H. Louisell, A. Yariv, A.E. Siegman, Phys. Rev. **124**, 1646 (1961); D.N. Klyshko, Pis'ma Zh. Éksp. Teor. Fiz. **9**, 69 (1969) [JETP Lett. **9**, 40 (1969)]; D.C. Burham, D.L. Weinberg, Phys. Rev. Lett. **25**, 84 (1970); Z.Y. Ou, L.J. Wang, L. Mandel, Phys. Rev. A **40**, 1428 (1989)
12. M. Fiorentino, P.L. Voss, X. Li, J.E. Sharping, G.A. Barbosa, P. Kumar, *Quantum Electronics and Laser Science Conference*, OSA Technical Digest, 2002, Vol. 74, pp. QPD1-3
13. J. Courtial, K. Dholakia, L. Allen, M.J. Padgett, Phys. Rev. A **56**, 4193 (1997)
14. N.B. Simpson, K. Dholakia, L. Allen, M.J. Padgett, Opt. Lett. **22**, 52 (1997)
15. M. Kristensen, J.P. Woerdman, Phys. Rev. Lett. **72**, 2171 (1994)
16. J.W.R. Tabosa, D.V. Petrov, Phys. Rev. Lett. **83**, 4967 (1999)
17. H.H. Arnaut, G.A. Barbosa, Phys. Rev. Lett. **86**, 5209 (2001)
18. G.A. Barbosa, H.H. Arnaut, Phys. Rev. A **65**, 053801 (2002)
19. M. Padgett, J. Courtial, L. Allen, S. Franke-Arnold, S.M. Barnett, J. Mod. Opt. **49**, 777 (2002)
20. J. Leach, M.J. Padgett, S.M. Barnett, S. Franke-Arnold, J. Courtial, Phys. Rev. Lett. **88**, 257901 (2002)
21. W.P. Gryce, I.A. Walmsley, Phys. Rev. A **56**, 1627 (1997)
22. S. Franke-Arnold, S.M. Barnett, M.J. Pagett, L. Allen, Phys. Rev. A **65**, 033823-1 (2002)
23. E.R. Eliel, S.M. Dutra, G. Nienhuis, J.P. Woerdman, Phys. Rev. Lett. **86**, 5208 (2001)
24. J. Courtial, K. Dholakia, D.A. Robertson, L. Allen, M.J. Padgett, Phys. Rev. Lett. **80**, 3217 (1998); J. Courtial, D.A. Robertson, K. Dholakia, L. Allen, M.J. Padgett, Phys. Rev. Lett. **81**, 4828 (1998)
25. A. Garuccio, Ann. New York Acad. Sci. **755**, 633 (1995)
26. P.G. Kwiat, P.H. Eberhard, A.M. Steinberg, R.Y. Chiao, Phys. Rev. A **49**, 3209 (1994); P.G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, Phys. Rev. Lett. **75**, 4337 (1995)
27. K. Mattle, H. Weinfurter, P.G. Kwiat, A. Zeilinger, Phys. Rev. Lett. **76**, 4656 (1996); G.A. Barbosa, Phys. Rev. A **58**, 3332 (1998)
28. Y.H. Shih, C.O. Alley, Phys. Rev. Lett. **61**, 2921 (1988)
29. G. Molina-Terriza, J.P. Torres, L. Torner, Phys. Rev. Lett. **88**, 013601 (2002)
30. R. Simon, G.S. Agarwal, Opt. Lett. **25**, 1313 (2000)